
# Latin cubes, Kronecker coefficients, and tensor invariants

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# 1. Latin squares

# Latin squares (for AI)

# Latin squares (for us)

- n x n square filled with 1,...,n
- Each row & column is a permutation

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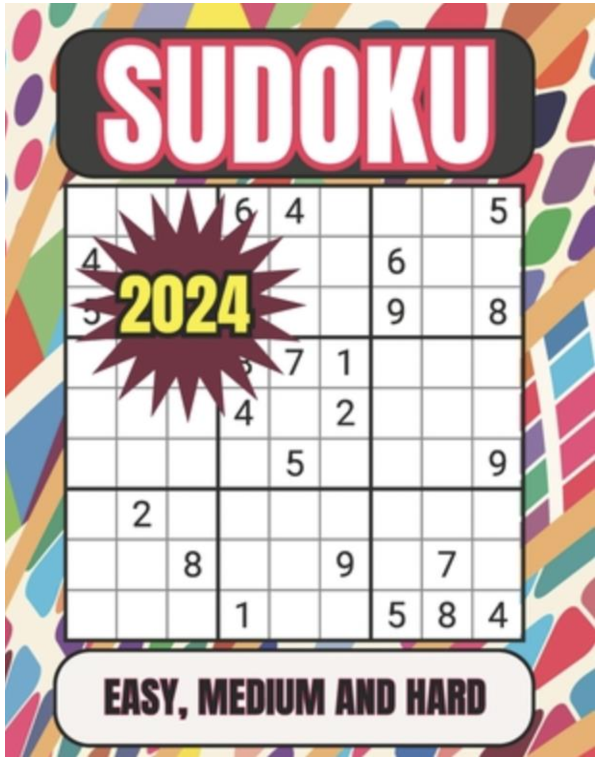


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# Sign of Latin square

$$\operatorname{sgn}(L) := \prod_{\text{rows } \sigma} \operatorname{sgn}(\sigma) \times \prod_{\text{cols } \sigma} \operatorname{sgn}(\sigma)$$

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Ex: sgn(L) = -1

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- AT motivation: Dinitz conjecture (proved later by Galvin '94)
- AT implies Rota's basis conjecture [Huang-Rota '94, Onn '97]
- AT holds for n=p+1, p-1 for primes p [Drisko '97, Glynn '10]

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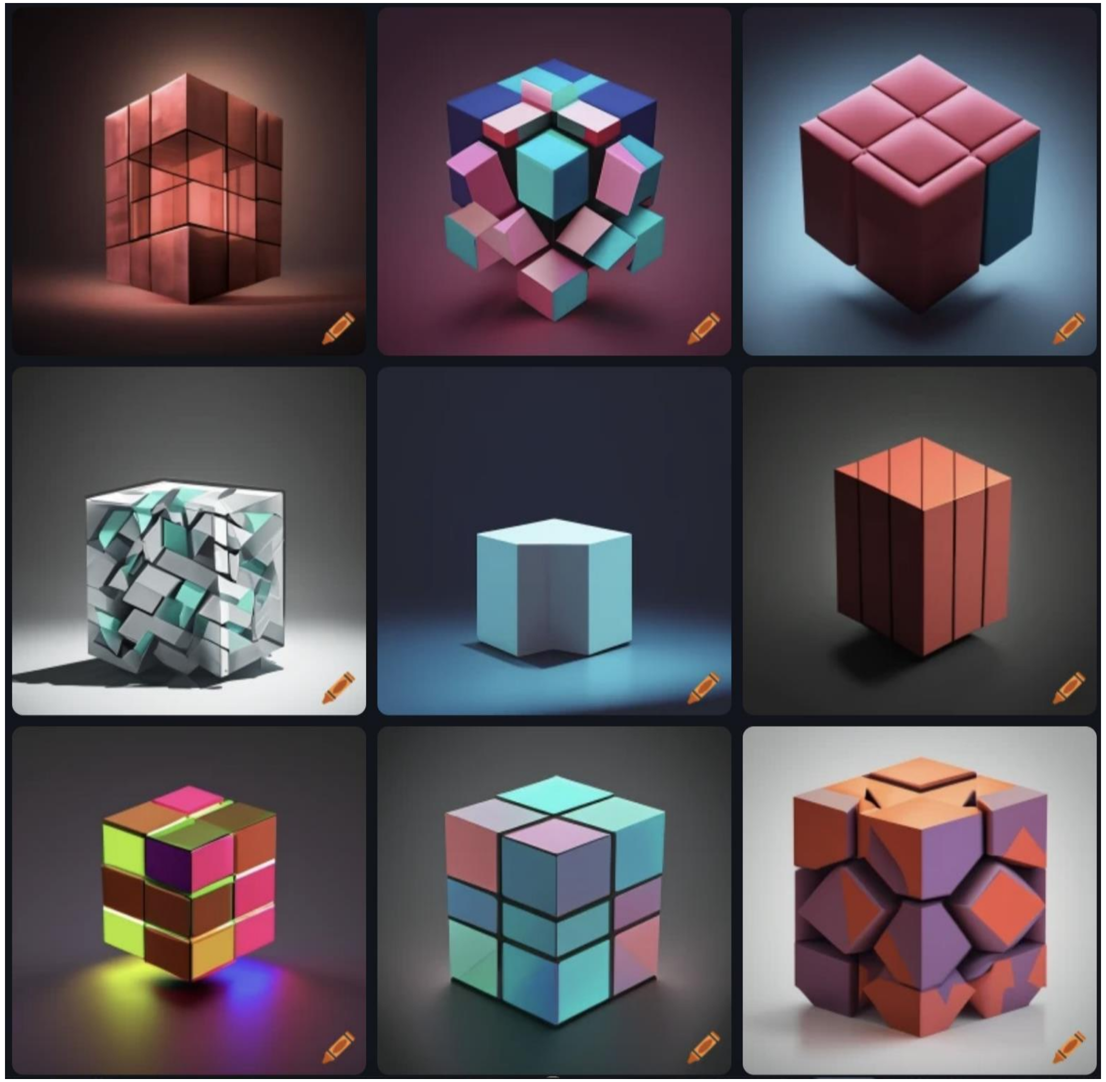
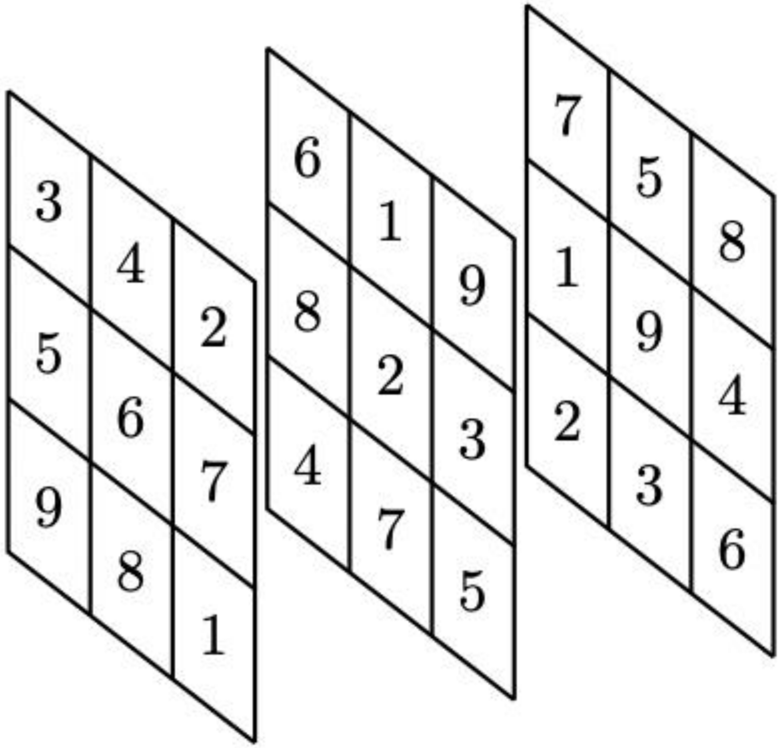
# Latin cubes (for AI)



# Latin cubes (for me)

- k x k x k cube filled with 1,...,$k^2$
- Each **slice** is a permutation of 1,...,$k^2$



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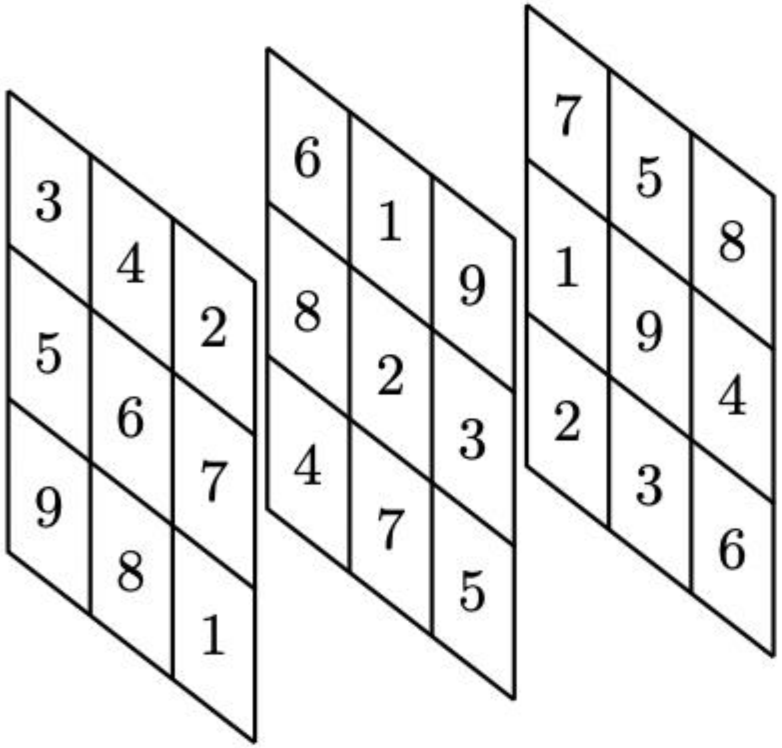


$$\operatorname{sgn}(L) := \prod_{\sigma \text{ slices in dir. } 1} \operatorname{sgn}(\sigma) \times \prod_{\sigma \text{ slices in dir. } 2} \operatorname{sgn}(\sigma) \times \prod_{\sigma \text{ slices in dir. } 3} \operatorname{sgn}(\sigma)$$

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*Motivation*: Geometric Complexity Theory

Fundamental SL-invariant (like 3d determinant) at unit cubic tensor

# 3. Kronecker coefficients

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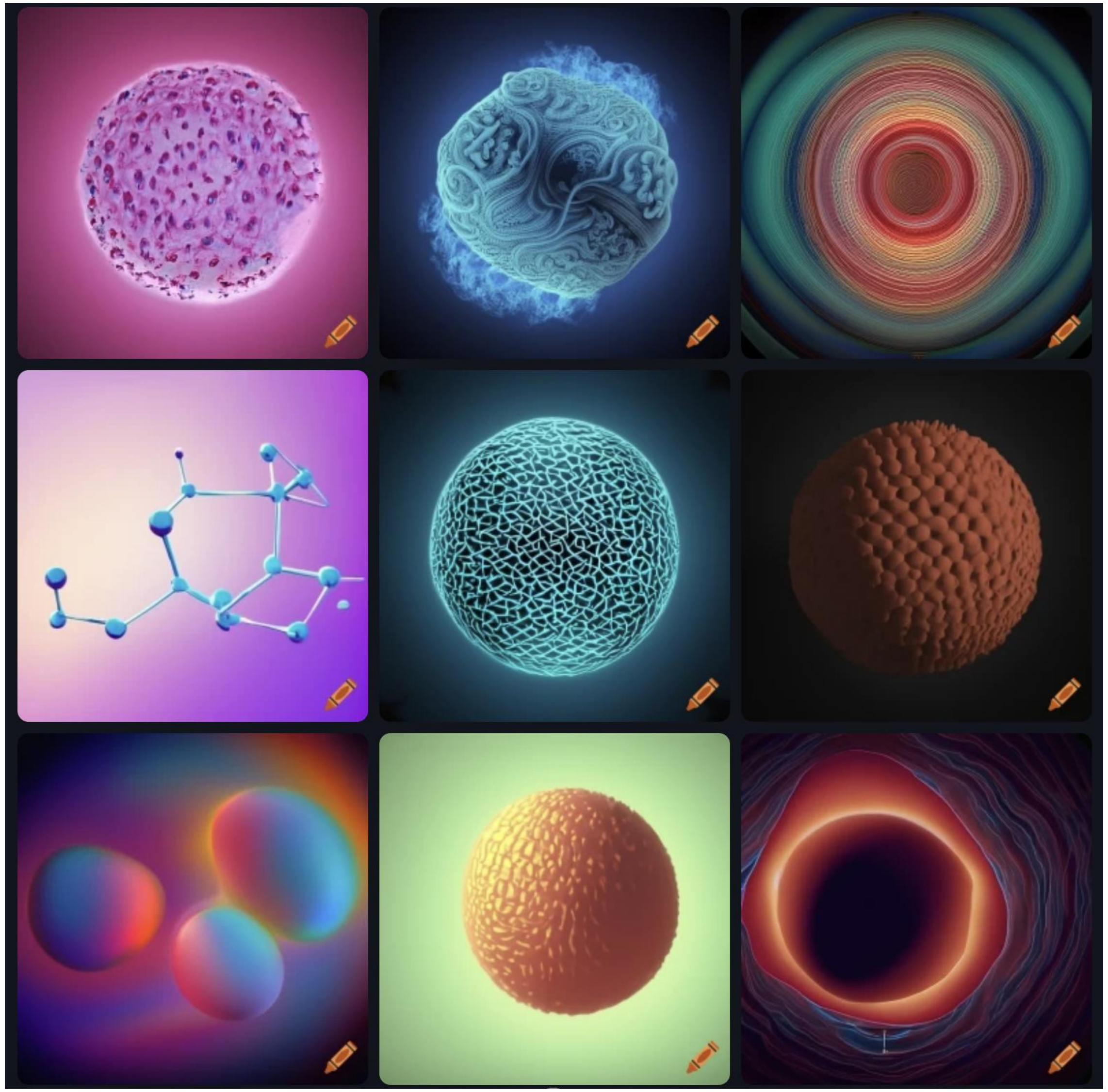

# Kronecker coefficients (for you)

$g(\lambda, \mu, \nu) =$ mult. $S^\lambda$ in $S^\mu \otimes S^\nu$, where $S^\alpha$ is irreducible representation (Specht module) of $S_n$ indexed by partition $\alpha$

$$s_\lambda(x_i y_j) = \sum_{\mu,\nu} g(\lambda, \mu, \nu) s_\mu(x_i) s_\nu(y_j)$$

*Schur polynomials* $s_\lambda(x_1, \ldots, x_n) := \dfrac{\det[x_i^{\lambda_j + n - j}]}{\prod_{i<j}(x_i - x_j)}$

- Important in Algebraic Combinatorics and Representation Theory
- No combinatorial interpretation known
- Deciding positivity is NP-hard [Ikenmeyer-Mulmuley-Walter, 2017]

# Rectangular Kronecker coefficients

**Fix** ***k*** and for $n$ = 0, 1, 2, ... consider the sequence:

$$g_n := g(k^n, k^n, k^n) \qquad \dim \mathbb{C}[(\mathbb{C}^n)^{\otimes 3}]_{nk}^{\mathrm{SL}_n^{\times 3}}$$

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Basic properties: $g_n = 0$ for $n > k^2$ and

symmetry: $g_n = g_{k^2-n}$

# Conjectures

$$g_n := g(k^n, k^n, k^n)$$

Fix *k* even.

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Note: $k = 3$, $g_n$ = 1, 1, 0, 1, 1, 1, 1, 0, 1, 1

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Special cases: $n < 4$, $n = k$ [Tewari, Bürgisser-Ikenmeyer, Bessenrodt-Behns]

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- Proofs rely on SL-invariants of tensors
- We also prove for generalized Kroneckers (of many partitions)

# Unimodality of 'qubit' Kroneckers

$$\bigwedge \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2$$

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Ex: For $d = 5$ and $n = 0, \ldots, 2^{d-1}$ we have

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**Theorem** (AY, 2023). $\{\beta_n\}_{n=0,\ldots,2^{d-1}}$ is unimodal.

Proof: Hard Lefschetz theorem on associated highest weight spaces of d-qubits


THE END